NORMANHURST BOYS HIGH SCHOOL

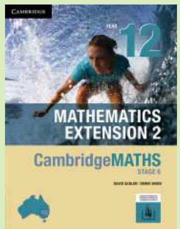
MATHEMATICS EXTENSION 2

Topic summary and exercises:

X2 **The Nature of Proof**

X2 **Inequalities**

With references to



Name: ....

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Initial version for Part II by H. Lam, July 2014. Updated April 2020 for new Mathematics Extension 2 syllabus.
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Various corrections by students & members of the Mathematics Department at Normanhurst Boys High School, as well as G. Sinclair at PLC Sydney

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Ⓐ Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argumen

Syllabus subtopics

MEX-P1 The Nature of Proof

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Extension 2* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

The Nature of Proof

Section 1

The Language of Proof

Learning Goal(s)

Knowledge
Vocabulary of proof

Skills
Identify type of proof required

Understanding
Differences between logical operators

By the end of this section am I able to:

20.1 Use the formal language of proof, including the terms statement, implication, converse, negation and contrapositive

1.1 Introduction

This section introduces the necessary **language** for proof.

Important note

- Beware of the mathematics teacher morphing into an English teacher for this section!
- Vocabulary, grammar are all important!

Fill in the spaces

A mathematical .... is an argument which convinces other people that something is true. In mathematics, we study .... , sentences that are either true or false but not both.

Example '8 is an even integer' and '4 is an odd integer' are statements.

Definition 1

Logical values A statement must take one of two logical values:

- .... , or
- ....

Example 1

Consider the statement

n is a multiple of 3

1. Can the logical value of this statement change with circumstance?

2. Can it be simultaneously both true and false?

Definition 2

A proven statement A statement which is shown to be .... is said to be .... , and the evidence used to establish the truth is called the ....

Counterexample A statement may be shown to be .... by a single .... , called a counterexample.

1.2 Logical operations: an introduction to predicate logic

- In arithmetic, numbers can be combined or modified with operations such as '+', '−', '×' and '÷', etc.
- Likewise, in logic, there are operations for combining and modifying statements, some of these operations are
  - '.........'
  - '..........',
  - '.....'
  - '.....................'

1.2.1 And, Or and Negation

Definition 3

Negation If p is a statement, then the statement 'not p' is called the .... of p.
"Not p" ($\neg p$, or $\sim p$) is defined to be

- true, whenever p is ....
- false, whenever p is ....

Definition 4

And If p and q are two statements, then the statement 'p and q' is defined to be

- true, when both p and q are ....
- false, when p is .... or q is .... or both p and q are ....

- Later at university: $p \wedge q$
- Set notation analogue: $P \cap Q$

Definition 5

Or If p and q are two statements, then the statement 'p or q' is defined to be

- true, when p is .... or q is .... or both p and q are ....
- false, when both p and q are ....

- Later at university: $p \vee q$
- Set notation analogue: $P \cup Q$

Fill in the spaces

And, Or and Negation The rules analogous to the complements of intersections of sets and unions of sets. Let A and B two sets.

- The negation of *and* is ....,

$$\neg(A \cap B) = \neg A \ldots \neg B$$

- The negation of *or* is ...., i.e. $\neg(A \text{ or } B) = \neg A$ .... $\neg B$

Steps

Verify that set theory/notation is entirely analogous to predicate logic with the following example:

> Let A be the set of Fibonacci numbers and B be the set of multiples of 3 inside the universal set U, the set of numbers on a die.

1. Draw a Venn diagram and find $A \cap B$ and $A \cup B$.

2. Verify: $\neg(A \cap B) =$ .... is represented on the Venn Diagram by shading

$$\overline{A \cap B}$$

3. Verify: $\neg(A \cup B) =$ .... is represented on the Venn Diagram by shading

$$\overline{A \cup B}$$

Example 2

[Ex 2A] Write down the negation of each statement.

(a) All cars are red.

(b) Hillary likes steak and pizza.

(c) If I live in Tasmania, then I live in Australia.

(d) $-3 \leq x \leq 8$

1.2.2 Implication

Important note

If p and q are two statements, then the statement 'if p then q' is abbreviated to

'p .... q' or $p \Rightarrow q$.

Example 3

[Ex 2A] Suppose that p is the statement *Jack does Mathematics Extension 2* and q is the statement *Jack is crazy.* Write each of the following as English sentences.

(a) $p \Rightarrow q$

(b) $\neg(p \Rightarrow q)$

(c) $\neg p \Rightarrow q$

(d) $\neg q \Rightarrow p$

(e) $\neg p \Rightarrow \neg q$

(f) $p \not\Rightarrow q$

1.3 Quantifiers

Consider the sentence 'x is even'. At this stage it's not possible to conclude whether it is true or false, as the value of x is still an unknown. There are three basic ways to turn this sentence into a statement.

- When x is 6, x is even.
- **For all** integers x, x is even.
- **There exists** an integer x such that x is even.

Definition 6

The phrases *for all* and *there exists* are called ....

Fill in the spaces

Notation for quantifiers

- $\forall$ means ....
- $\exists$ means ....

Example 4

Rewrite the following statements with the notation of quantifiers.

(a) For all integers n, the integer $n(n+1)$ is even.

(b) There exists an integer n such that $n^2 - n + 1 = 0$.

Example 5

Write each statement as an English sentence, without any use of symbols.

(a) $\forall n \in \mathbb{Z}$, $\exists\, m \in \mathbb{Z}$ such that $m > n$

(b) $a \in \mathbb{R}$ and $a > 0 \Rightarrow a + \dfrac{1}{a} \geq 2$

1.4 More about statements

1.4.1 Sufficient and Necessary

Fill in the spaces

Sufficiency and necessary condition In the statement

If p then q,

- p is a .... condition for q
- q is a .... condition for p

Example 6

Consider the following statements.

(a) If I travel by bus then I use public transport.

(b) If a number on a die is prime then it is a Fibonacci number.

Verify their sufficiency and necessary condition.

1.4.2 Converse statements

Definition 7

Converse statements The .... of the statement *if* p *then* q is the statement *if* q *then* p.
i.e. The .... of the implication $p \Rightarrow q$ is ....

Example 7

Write down the converse of the following statements.

(a) If x is a multiple of 4 then x is even.

(b) If a quadrilateral is a rhombus then it has four equal sides.

Does a statement always have the same logical value as its converse?

1.4.3 Equivalent statements

Definition 8

Equivalent statements Two statements are equivalent if each is a .... of the other.
i.e. Two statements p and q are equivalent if $p \Rightarrow q$ and $q \Rightarrow p$ have the same ....

Important note

- The two implications can be abbreviated into one statement using the phrase

  '..... ....'

- Spelt in shorthand: ....
- Notation: $\Leftrightarrow$.

Fill in the spaces

Thus, the geometry example from Example 7 on the facing page can be written as

- A quadrilateral is a rhombus if and only if it has four equal sides.
  or
- A quadrilateral is a rhombus .... it has four equal sides.
  or
- A quadrilateral is a rhombus .... if it has four equal sides.

Important note

When two statements are equivalent, each is both a .... and .... condition for the other to be true.

Example 8

Consider the following true statement:

If x is a multiple of 4 then x is even.

This is **logically equivalent** to the statement

If x is not even then x is not a multiple of 4.

What would be the logically equivalent statement to the following statement:

If I own a dog then I have a pet.

1.4.4 The Contrapositive

Definition 9

Contrapositive For two statements p and q, suppose that $p \Rightarrow q$.

- The .... of the implication is

  ....

- An implication and its contrapositive are always ....

Fill in the spaces

An implication and its contrapositive can be written in one statement:

A implies B .... B implies .... A

Write this entirely in symbols.

Further exercises

(X2) **Ex 2A**

- All questions

1.4.5 Truth tables and Boolean algebra

Truth tables can assist with 'visualising' the the logical results previously shown. Some additional reading on medium.com

Important note

Not in the Extension 2 syllabus, but an understanding of this subsection will help immensely with the previously covered material.

Negation

Fill in the spaces

p	$\neg p$
T	
F	

p	$\neg p$
1	
0	

And

Fill in the spaces

p	q	$p \wedge q$
T	T	
F	T	
T	F	
F	F	

p	q	$p \wedge q$
0	0	
1	0	
0	1	
1	1	

Or

Fill in the spaces

p	q	$p \vee q$
T	T	
F	T	
T	F	
F	F	

p	q	$p \vee q$
0	0	
1	0	
0	1	
1	1	

Implication/Contrapositive

Fill in the spaces

p	q	$p \Rightarrow q$
T	T	
F	T	
T	F	
F	F	

p	q	$p \Rightarrow q$
0	0	
1	0	
0	1	
1	1	

p	q	$\neg p$	$\neg p \vee q$
T	T		
F	T		
T	F		
F	F		

p	q	$\neg p$	$\neg p \vee q$
0	0		
1	0		
0	1		
1	1		

Iff

By definition,

$$p \Leftrightarrow q = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Fill in the spaces

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T			
F	T			
T	F			
F	F			

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
0	0			
1	0			
0	1			
1	1			

History

George Boole (1815-1864) was a largely self-taught English mathematician, philosopher and logician. He worked in the fields of differential equations and algebraic logic, and is best known as the author of *The Laws of Thought* (1854) which contains *Boolean algebra*. Boolean logic is credited with laying the foundations for the information age. **Source:** Wikipedia

Section 2

Number Proofs - Divisibility

Learning Goal(s)

Knowledge	Skills	Understanding
Divisibility results	Completing numerical proofs	When to use direct versus counterexample to prove results

By the end of this section am I able to:

20.2 Prove simple results involving numbers

Review of number systems

- $\mathbb{Z}$ ....
- $\mathbb{Z}^+$ ....
- $\mathbb{N}$ ....

Fill in the spaces

Let $a, b, m \in \mathbb{Z}$ and suppose that $b = am$.

- The numbers a and m are called .... of b.
- b is said to be .... by a and m.

Theorem 1

Divisibility If $a, b \in \mathbb{Z}$ and b is divisible by a then $\exists m \in \mathbb{Z}$ such that $b = am$.

Example 9

[**Ex 2B**] Let a and b be two integers divisible by 3.

(a) Prove that $(a + b)$ is divisible by 3.

(b) Prove that $(ax + by)$ is divisible by 3, $\forall x, y \in \mathbb{Z}$

Example 10

[**Ex 2B**] Prove that $a^2 - a$ is even $\forall a \in \mathbb{Z}$.

Example 11

[**Ex 2B**] A student claims that if an integer n is divisible by both 4 and 6 then the number is divisible by $4 \times 6 = 24$.

(a) Disprove this claim by finding a counterexample.

(b) Explain what has gone wrong, and determine the correct conclusion.

Example 12

[Ex 2B] Let $n = 10x + y$, where $n, x, y \in \mathbb{Z}^+$.

(a) Prove that if n is divisible by 7 then $(x - 2y)$ is also divisible by 7.

(b) Further, prove that the converse is true.

(c) Write the result as an iff statement.

(d) Hence determine whether or not 3871 is divisible by 7.

Example 13

[Ex 2B]

(a) Prove that the square of an even number is divisible by 4.

(b) Prove that the remainder is 1 when the square of an odd number is divided by 8.

(c) Hence prove that if a and b are both odd, then $a^2 + b^2$ is not a square.

Example 14

[2020 NBHS Mathematics Ext 2 Assessment Task 3] (3 marks) Prove that

> A three digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Hint: Let the digits be a, b and c.

Further exercises

Ex 2B

- All questions

Section 3

Proof by Contraposition and Contradiction

Learning Goal(s)

Knowledge
Contrapositive statements

Skills
Identifying when to prove by contrapositive or contradiction

Understanding
Difference between direct and contrapositive statements

By the end of this section am I able to:

20.3 Use proof by contradiction including proving the irrationality for numbers such as $\sqrt{2}$ and $\log_2 5$.

20.4 Use examples and counter-examples

- *Proof by contraposition* and *proof by contradiction* are commonly used in mathematics and involve negation.

3.1 Proof by Contraposition

Important note

- **Reminder!**
  An implication is equivalent to its ....

- When an implication is not easy to prove directly, it may be suitable to use proof by .... instead. It is important to *clearly* state what is being done at the ....

Example 15

[**Ex 2C**] Prove that if n^2 is even then n is even.

Example 16

[2020 NBHS Mathematics Ext 2 Assessment Task 3] (2 marks) Prove the following statement by contrapositive:

For $n \in \mathbb{Z}$, if $n^2 - 6n + 5$ is even, then n is odd.

3.2 Proof by Contradiction

Important note

- **Reminder!**
  - A statement can only have one of two values, ... or ....
  - The negated statement must have the .... value.
- When an a statement is not easy to prove directly, it may be suitable to show instead that the .... statement is ....
- Begin by writing down the .... and show this leads to a ....

Example 17

[Ex 2C] Prove that $\sqrt{2}$ is irrational.

Example 18

[**Ex 2C**] Prove that $\log_2 3$ is irrational.

Example 19

Prove that if $2^n - 1$ is prime then n is prime by proving the contrapositive.

Example 20

[Ex 2C] Suppose that $a, b, c, d \in \mathbb{Z}^+$ and c is not a square.

(a) Prove that if $\dfrac{a}{b+\sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is rational, then $b^2 d = c(a+d)$.

(b) Hence prove by contradiction that $\dfrac{a}{1+\sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is irrational.

Further exercises

(X2) **Ex 2C**

- All questions

3.3 HSC questions

Example 21

[2020 Ext 2 HSC Q7] Consider the proposition:

If $2^n - 1$ is not prime, then n is not prime.

Given that each of the following statements is true, which statement disproves the proposition?

(A) $2^5 - 1$ is prime

(B) $2^6 - 1$ is divisible by 9

(C) $2^7 - 1$ is prime

(D) $2^{11} - 1$ is divisible by 23

Example 22

[2020 Ext 2 HSC Q8] Consider the statement:

If n is even, then if n is a multiple of 3, then n is a multiple of 6.

Which of the following is the negation of this statement?

(A) n is odd and n is not a multiple of 3 or 6.

(B) n is even and n is a multiple of 3 but not a multiple of 6.

(C) If n is even, then n is not a multiple of 3 and n is not a multiple of 6.

(D) If n is odd, then if n is not a multiple of 3 then n is not a multiple of 6.

Example 23

[2020 Ext 2 HSC Q14] (3 marks) Prove that for any integer $n > 1$, $\log_n(n+1)$ is irrational.

Example 24

[**2020 Ext 2 HSC Q15**] In the set of integers, let P be the proposition:

If $k+1$ is divisible by 3, then k^3+1 is divisible by 3.

i. Prove that the proposition P is true. **2**

ii. Write down the contrapositive of the proposition P. **1**

iii. Write down the converse of the proposition P and state, with reasons, whether this converse is true or false. **3**

Part II

Inequalities

Section 4

Algebraic inequalities

Learning Goal(s)

Knowledge
Arithmetic-Geometric Means

Skills
Prove algebraic results based on general properties of numbers and the AM-GM inequality

Understanding
When to use the AM-GM inequality

By the end of this section am I able to:
25.1 Prove results involving inequalities.

4.1 General results

All questions hinge on one or more of the following results:

Theorem 2

If a, b, c and $d \in \mathbb{R}$,

1. Either:

(a) $a > b$ (b) $a = b$, or (c) $a < b$

2. If $a > b$, then ....

3. If $a > 0$ and $b > 0$, then ....

4. If $a > 0$ and $b > 0$, then ....

5. Perfect squares: $(a-b)^2 \geq 0$, always. $(a-b)^2 = 0$ iff ....

6. Perfect squares: $(a+b)^2 \geq 0$, always. $(a+b)^2 = 0$ iff ....

7. Multiplication by positive constant: If $a \geq b$ and $c > 0$, then ....

8. Multiplication by negative constant: If $a \geq b$ and $c < 0$, then ....

9. Transitivity: If $a \geq b$, $b \geq c$ and $c \geq d$, then ....

10. Addition: If $a \leq b$ and $c \leq d$, then ....

11. Addition: If $a \geq b$ and $c \geq d$, then ....

12. Multiplication: If $a \geq b > 0$ and $c \geq d > 0$, then .... and ....

(These results may be quoted without proof.)

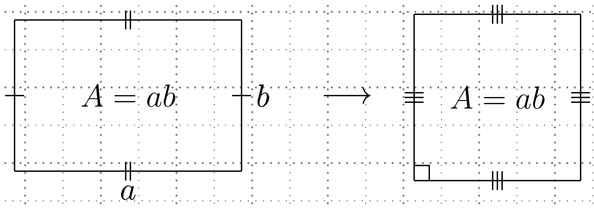
4.2 AM-GM inequality

Definition 10

If $a,\ b \in \mathbb{R}^+$,

- **Arithmetic mean:** $\dfrac{a+b}{2}$.
- **Geometric mean:** $\sqrt{ab}$.

- The geometric mean is equivalent side length of a square (of equal area) given a rectangle of dimensions a and b.



Theorem 3

Arithmetic Mean-Geometric Mean (AM-GM) Inequality

$$\frac{a+b}{2} \geq \sqrt{ab} \quad a, b \in \mathbb{R}^+$$

Proof

Steps

1. Expand $(x-y)^2 \geq 0$:

2. Add $4xy$ to both sides:

3. Factorise, and divide:

4.2.1 Enumerating multiplied terms

Example 25

[**Ex 2D Q6**] If a, b and $c \in \mathbb{R}^+$, prove that $(a+b)(a+c)(b+c) \geq 8abc$.

Prove AM-GM inequality, then enumerate

4.2.2 Enumerating added terms

Example 26

Prove $a^2 + b^2 + c^2 \geq ab + bc + ac$, $\forall a, b, c \in \mathbb{R}$.

Example 27

[Ex 2D Q7] Suppose p, q and r are real and distinct.

(a) Prove that $p^2 + q^2 > 2pq$.

(b) Hence prove that $p^2 + q^2 + r^2 > pq + qr + rp$.

(c) Given that $p + q + r = 1$, prove that $pq + qr + rp < \dfrac{1}{3}$.

Example 28

[2011 Ext 2 HSC Q5] (3 marks) If p, q and $r \in \mathbb{R}^+$, and $p + q \geq r$, prove that

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0$$

Important note

Hint 'Brute force' your way through this.

Example 29

(a) Show that if $a > 0$, $a + \dfrac{1}{a} \geq 2$.

(b) Deduce for a, b, $c \in \mathbb{R}^+$, then

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$$

and

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

(c) Hence show that

$$\frac{9}{a+b+c} \leq \frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and state the conditions under which equality holds.

Example 30

[Ex 2D Q9] Suppose a, b and $c > 0$.

(a) Prove $a^2 + b^2 \geq 2ab$.

(b) Hence prove $a^2 + b^2 + c^2 \geq ab + bc + ac$.

(c) Given that $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$, then show that

$$a^3 + b^3 + c^3 \geq 3abc$$

(d) Hence show that $x + y + z \geq 3\sqrt[3]{xyz}$ for x, y and $z \in \mathbb{R}^+$.

Example 31

[2012 CSSA Ext 2 Trial Q16]

i. The inequality $\dfrac{x+y}{2} \geq \sqrt{xy}$ is true for all $x \geq 0$ and $y \geq 0$, with equality when $x = y$. **1**

Use the inequality to show that the minimum value of the function $f(x) = Ae^x + Be^{-x}$ is $2\sqrt{AB}$, where $A > 0$ and $B > 0$ are constants.

ii. The function $f(x) = Ae^x + Be^{-x}$ has line symmetry about the vertical line $x = c$. That is, we can write $f(x)$ in the form $f(x) = k\left[e^{x-c} + e^{-(x-c)}\right]$ for some values of k and c. **2**

Using part (i), or otherwise, find k and c in terms of A and B.

Answer: $k = \sqrt{AB}$, $c = \ln\sqrt{\frac{B}{A}}$

4.2.3 Combination of inequalities

- Mostly involve showing a particular function is always .... or .... within a certain interval.

Example 32

[2013 Independent Ext 2 Trial Q15] Consider the function

$$f(x) = \sum_{k=1}^{n} \left(\sqrt{a_k}\, x - \frac{1}{\sqrt{a_k}} \right)^2$$

where $a_1,\, a_2,\, \cdots,\, a_n \in \mathbb{R}^+$.

i. By expressing $f(x)$ as a quadratic function of x, show that **3**

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$$

ii. Hence show that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} \geq \dfrac{2n}{n+1}$. **2**

Example 33

[2012 Ext 2 HSC Q15] (Also in Sadler and Ward (2019))

i. Prove that $\sqrt{ab} \leq \dfrac{a+b}{2}$, where $a > 0$ and $b \geq 0$. **1**

ii. If $1 \leq x \leq y$, show that $x(y - x + 1) \geq y$. **2**

iii. Let n and j be positive integers with $1 \leq j \leq n$. Prove that **2**

$$\sqrt{n} \leq \sqrt{j(n-j+1)} \leq \frac{n+1}{2}$$

iv. For integers $n \geq 1$, prove that **1**

$$\left(\sqrt{n}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

Further exercises

Ex 2D

- Q1-18

Section 5

Inequalities in Geometry and Calculus

Learning Goal(s)

Knowledge
Using geometric facts (e.g. triangle inequality), as well as first/second derivatives and integrals to prove results

Skills
Identify when to use first and/or second derivative or integrals to prove results

Understanding
More properties of functions in relation to their derivatives and integrals

By the end of this section am I able to:

25.2 Prove further results involving inequalities by logical use of previously obtained inequalities

Important note

This section contains multidisciplinary questions, often mixing the inequalities work within **MEX-P1 The Nature of Proof** with any Mathematics Advanced or Extension 1 content.

5.1 Inequalities via graphical methods

Example 34

Show that $x \geq \ln(1+x)$, $\forall x > -1$. State when equality holds.

Example 35

Show that $\cos x \geq 1 - \dfrac{1}{2}x^2$, $\forall x$.

5.2 Triangle inequality

5.2.1 Forward inequality

Theorem 4

The triangle inequality $\forall a, b \in \mathbb{R}$:

$$|a+b| \leq \text{................}$$

Steps

Proof Uses $|x|^2 = x^2$

1. Square $(|a| + |b|)$:

2. Note that $|a|\,|b| \geq ab$:

3. Remove squares:

5.2.2 Reverse inequality

Theorem 5

The 'reverse' triangle inequality $\forall a, b \in \mathbb{R}$:

$$|a+b| \geq \ldots\ldots\ldots\ldots\ldots\ldots$$

Steps

Proof Uses Theorem 4 on the preceding page.

1. Rewrite $|a| = |(a+b) - b|$ and use triangle inequality:

$$|a| - |b| \leq |a+b|$$

2. Do likewise to $|b|$:

$$|b| - |a| \leq |a+b|$$

3. Use the definition of the absolute value, i.e.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

(replace x with $|a| - |b|$)

5.3 Inequalities by differentiation

Example 36

[2013 NSGHS Ext 2 Trial Q16] Consider $f(x) = \log x - x + 1$.

i. Show that $f(x) \leq 0$ for all $x > 0$. **2**

ii. Consider the set of n positive numbers p_1, p_2, p_3 $\ldots p_n$ such that **1**

$$p_1 + p_2 + p_3 + \cdots + p_n = 1$$

By using the result in part (i), show that

$$\sum_{r=1}^{n} \log(np_r) \leq np_1 + np_2 + np_3 + \cdots + np_n - n$$

iii. Show that $\sum_{r=1}^{n} \log(np_r) \leq 0$. **1**

iv. Hence deduce that $0 < n^n p_1 p_2 p_3 \cdots p_n \leq 1$. **1**

Example 37

[2019 Independent Ext 2 Trial Q13]

i. If $f(x) = x - \ln\left(\dfrac{\cos x}{1 - \sin x}\right)$ for $0 \leq x \leq \frac{\pi}{2}$, show that **2**

$$f'(x) = 1 - \sec x$$

ii. Hence show that $e^x < \dfrac{\cos x}{1 - \sin x}$ for $0 < x < \frac{\pi}{2}$. **3**

5.4 Inequalities by integration

See also: **Topic 27 - X1 X2 Further Integration**.

Example 38

Use $\ln t = \int_1^t \frac{1}{x}\,dx$ for $t > 1$ to deduce that

$$1 - \frac{1}{t} \leq \ln t \leq \frac{1}{2}\left(t - \frac{1}{t}\right)$$

for $t \geq 1$.

Example 39

[1997 4U HSC Q6] The series $1 - x^2 + x^4 - \cdots + x^{4n}$ has $2n + 1$ terms.

(i) Explain why **1**

$$1 - x^2 + x^4 - \cdots + x^{4n} = \frac{1 + x^{4n+2}}{1 + x^2}$$

(ii) Hence show that **2**

$$\frac{1}{1 + x^2} \leq 1 - x^2 + x^4 - \cdots + x^{4n} \leq \frac{1}{1 + x^2} + x^{4n+2}$$

(iii) Hence show that, if $0 \leq y \leq 1$, then **3**

$$\tan^{-1} y \leq y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots + \frac{y^{4n+1}}{4n + 1} \leq \tan^{-1} y + \frac{1}{4n + 3}$$

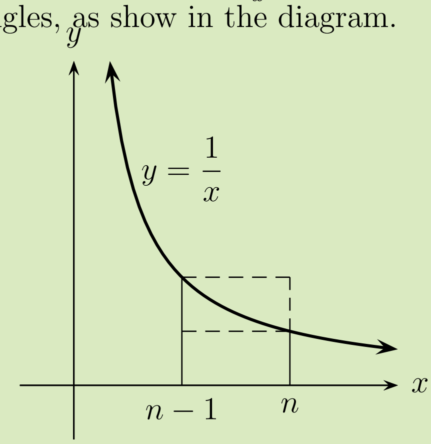
(iv) Deduce that **1**

$$0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{1001}\right) - \frac{\pi}{4} < 10^{-3}$$

Example 40

[2009 Ext 2 HSC Q8] Let $n \in \mathbb{Z}^+$, $n > 1$.

The area of the region under the curve $y = \frac{1}{x}$ from $x = n-1$ to $x = n$ is between the areas of the two rectangles, as show in the diagram.



Show that

$$e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$$

Further exercises

Ex 2D

- Q1-18

Further exercises (Legacy Textbooks)

Ex 8.1 Arnold and Arnold (2000)
Ex 8.2 Lee (2006)

Section A

Past HSC questions

- Questions in this appendix appear only if they have not yet appeared in prior booklets for **Topic 16 - X2 Complex Numbers** or **Topic 27 - X1 X2 Further Integration**.
- Questions earmarked ? indicates that it is uncertain whether a question of this type can appear in the new 2019-2020 syllabuses, given this *escape clause* in the new syllabuses:

  > Prove further results involving inequalities by logical use of previously obtained inequalities (*Mathematics Extension 2 Stage 6 Syllabus*, 2017, Revised 18/11/2019, p.28)

  It is uncertain due to one, or both of the following:

  - Level of difficulty - does it get this difficult?
  - Reach into other parts of the syllabuses - does it go this far outside of inequalities?

A.1 1998 4U HSC

Question 8

(a) The numbers p, q and s are fixed and positive. Also $p > 1$, $q > 1$ and $p = \dfrac{q}{q-1}$.

i. What positive value of t minimises the expression **2**

$$f(t) = \frac{s^p}{p} + \frac{t^q}{q} - st?$$

ii. Show that for all $t > 0$, **1**

$$\frac{s^p}{p} + \frac{t^q}{q} \geq st$$

iii. Prove by induction that **4**

$$(x_1 x_2 \times \cdots \times x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all $x_1, \ldots, x_n > 0$.

iv. Deduce that, for all $y_1, y_2, \ldots, y_n > 0$, **1**

$$\frac{y_1}{y_2} + \frac{y_2}{y_3} + \cdots + \frac{y_{n-1}}{y_n} + \frac{y_n}{y_1} \geq n$$

A.2 2000 4U HSC

Question 7

(a) i. Show that, for $x > 0$, $\ln x < x - 1$, with equality only at $x = 1$. **2**

ii. From (i), deduce that **3**

$$\sum_{i=1}^{n} x_i \ln \frac{y_i}{x_i} \leq 0$$

whenever $\sum_{i=1}^{n} x_1 = \sum_{i=1}^{n} y_1 = 1$, where $x_i > 0$, $y_1 > 0$ for $i = 1, 2, \ldots, n$.

Show that equality occurs only if $x_i = y_i$ for $i = 1, 2, \ldots n$.

iii. By considering part (ii) with equal values of y_1 for $i = 1, 2, \ldots, n$, prove that the maximum value of **3**

$$\sum_{i=1}^{n} x_1 \ln \frac{1}{x_i} = \ln n$$

where $\sum_{i=1}^{n} x_1 = 1$ and $x_i > 0$ for $i = 1, 2, \ldots n$.

iv. Does the result of part (iii) hold if ln is replaced by $\log_2$? Give reasons for your answer. **1**

A.3 2001 Ext 2 HSC

Question 8

(a) i. Show that $2ab \leq a^2 + b^2$ for all real numbers a and b. **3**

Hence deduce that $3(ab + bc + ca) \leq (a + b + c)^2$ for all real numbers a, b and c.

ii. Suppose a, b and c are sides of a triangle. Explain why $(b - c)^2 \leq a^2$. **4**

Deduce that $(a + b + c)^2 \leq 4(ab + bc + ca)$.

A.4 2003 Ext 2 HSC

Question 6

(c) i. Let x and y be real numbers such that $x \geq 0$ and $y \geq 0$. **1**

Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$

ii. Suppose that a, b c are real numbers. **2**

Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

iii. Show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$. **2**

iv. Deduce that if $a + b + c = d$, then $a^4 + b^4 + c^4 \geq abcd$. **1**

Question 7 ?

(c) Suppose that α is a real number with $0 < \alpha < \pi$.

Let $P_n = \cos\left(\dfrac{\alpha}{2}\right)\cos\left(\dfrac{\alpha}{4}\right)\cos\left(\dfrac{\alpha}{8}\right)\cdots\cos\left(\dfrac{\alpha}{2^n}\right)$.

i. Show that $P_n \sin\left(\dfrac{\alpha}{2^n}\right) = \dfrac{1}{2}P_{n-1}\sin\left(\dfrac{\alpha}{2^{n-1}}\right)$. **2**

ii. Deduce that $P_n = \dfrac{\sin\alpha}{2^n \sin\left(\dfrac{\alpha}{2^n}\right)}$. **1**

iii. Given that $\sin x < x$ for $x > 0$, show that **2**

$$\frac{\sin\alpha}{\cos\left(\dfrac{\alpha}{2}\right)\cos\left(\dfrac{\alpha}{4}\right)\cos\left(\dfrac{\alpha}{8}\right)\cdots\cos\left(\dfrac{\alpha}{2^n}\right)} < \alpha$$

A.5 2004 Ext 2 HSC

Question 7

(a) i. Let a be a positive real number. Show that $a + \dfrac{1}{a} \geq 2$. **2**

ii. Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be n positive real numbers. **4**

Prove by induction that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$$

iii. Hence show that $\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \geq 9\cos^2\theta$ **1**

A.6 2005 Ext 2 HSC

Question 8

(a) Suppose that a and b are positive real numbers, and let $f(x) = \dfrac{a+b+x}{3(abx)^{\frac{1}{3}}}$.

i. Show that the minimum value of $f(x)$ occurs at $x = \dfrac{a+b}{2}$ **3**

ii. Suppose that c is a positive real number. **2**

Show that $\left(\dfrac{a+b+c}{3\sqrt[3]{abc}}\right)^3 \geq \left(\dfrac{a+b}{2\sqrt{ab}}\right)^2$ and deduce $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.

You may assume that $\dfrac{a+b}{2} \geq \sqrt{ab}$.

iii. Suppose that the cubic equation $x^3 - px^2 + qx - r = 0$ has three positive real roots. Use part (ii) to prove that $p^3 \geq 27r$. **1**

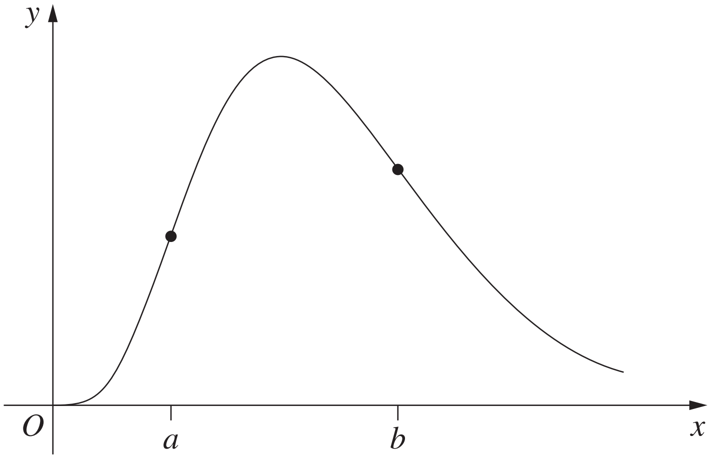
iv. Deduce that the cubic equation $x^3 - 2x^2 + x - 1 = 0$ has exactly one real root. **2**

A.7 2006 Ext 2 HSC

Question 8

(a) See the **Topic 27 - X1 X2 Further Integration** booklet for part (a), which contains prerequisites to part (b).

(b) For $x > 0$, let $f(x) = x^n e^{-x}$, where n is an integer and $n \geq 2$.



i. The two points of inflexion of $f(x)$ occur at $x = a$ and $x = b$, where $0 < a < b$. Find a and b in terms of n. **4**

ii. Show that **2**

$$\frac{f(b)}{f(a)} = \left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}}$$

iii. Using the result of part (a) (iv), show that **2**

$$1 \leq \frac{f(b)}{f(a)} \leq e^{\frac{4}{3\sqrt{n}}}$$

iv. What can be said about the ratio $\dfrac{f(b)}{f(a)}$ as $n \to \infty$? **1**

A.8 2007 Ext 2 HSC

Question 6

(a) i. Use the binomial theorem **1**

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \cdots + b^n$$

to show that, for $n \geq 2$,

$$2^n > \binom{n}{2}$$

ii. Hence show that, for $n \geq 2$, **2**

$$\frac{n+2}{2^{n-1}} < \frac{4n+8}{n(n-1)}$$

iii. Prove by induction that, for integers $n \geq 1$, **3**

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$$

Question 7

(a) i. Show that $\sin x < x$ for $x > 0$. **2**

ii. Let $f(x) = \sin x - x + \dfrac{x^3}{6}$. Show that the graph of $y = f(x)$ is concave up for $x > 0$. **2**

iii. By considering the first two derivatives of $f(x)$, show that $\sin x > x - \dfrac{x^3}{6}$ for $x > 0$. **2**

A.9 2009 Ext 2 HSC

Question 5

(a) Let $f(x) = \dfrac{e^x - e^{-x}}{2} - x$

i. Show that $f''(x) > 0$ for all $x > 0$. **2**

ii. Hence, or otherwise, show that $f'(x) > 0$ for all $x > 0$. **2**

iii. Hence, or otherwise, show that $\dfrac{e^x - e^{-x}}{2} > x$ for all $x > 0$. **1**

A.10 2010 Ext 2 HSC

Question 4

(c) Let k be a real number, $k \geq 4$. **3**

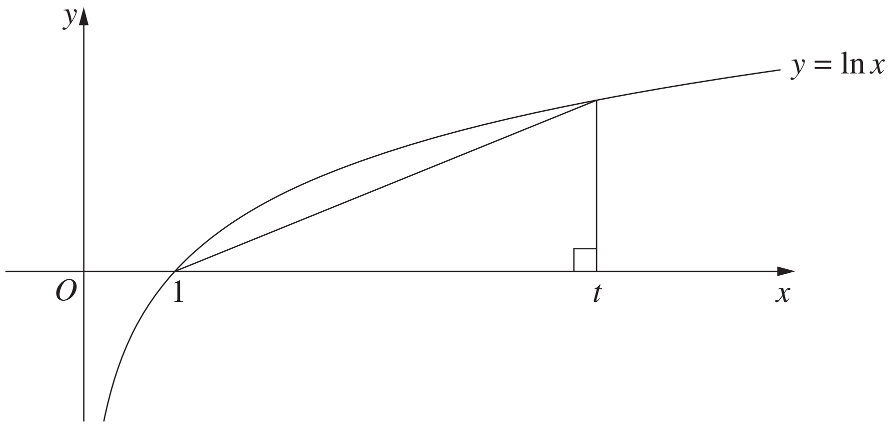
Show that, for every positive real number b, there is a positive real number a such that

$$\frac{1}{a} + \frac{1}{b} = \frac{k}{a+b}$$

A.11 2013 Ext 2 HSC

Question 14

(a) The diagram shows $y = \ln x$. **3**



By comparing relevant areas in the diagram, or otherwise, show that

$$\ln t > 2\left(\frac{t-1}{t+1}\right)$$

for $t > 1$

Question 16

(a) i. Find the minimum value of **2**

$$P(x) = 2x^3 - 15x^2 + 24x + 16 \qquad x \geq 0$$

ii. Hence, or otherwise, show that for $x \geq 0$, **1**

$$(x+1)\left(x^2 + (x+4)^2\right) \geq 25x^2$$

iii. Hence, or otherwise, show that for $m \geq 0$ and $n \geq 0$, **2**

$$(m+n)^2 + (m+n+4)^2 \geq \frac{100mn}{m+n+1}$$

A.12 2014 Ext 2 HSC

Question 15

(a) Three positive real numbers a, b and c are such that $a + b + c = 1$ and $a \leq b \leq c$. **2**

By considering the expansion of $(a+b+c)^2$, or otherwise, show that

$$5a^2 + 3b^2 + c^2 \leq 1$$

A.13 2015 Ext 2 HSC

Question 15

(b) Suppose that $x \geq 0$ and n is a positive integer.

i. Show that $\quad 1 - x \leq \dfrac{1}{1+x} \leq 1$ **2**

ii. Hence, or otherwise, show that **2**

$$1 - \frac{1}{2n} < n \ln\left(1 + \frac{1}{n}\right) \leq 1$$

iii. Hence, explain why $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ **1**

(c) For positive real numbers x and y, $\sqrt{xy} \leq \dfrac{x+y}{2}$ (Do NOT prove this).

i. Prove $\sqrt{xy} \leq \sqrt{\dfrac{x^2+y^2}{2}}$, for positive real numbers x and y. **1**

ii. Prove that $\sqrt[4]{abcd} \leq \sqrt{\dfrac{a^2+b^2+c^2+d^2}{4}}$, for positive real numbers a, b, c and d. **2**

A.14 2016 Ext 2 HSC

Question 14

(c) Show that $x\sqrt{x} + 1 \geq x + \sqrt{x}$, for $x \geq 0$. **3**

A.15 2018 Ext 2 HSC

⚠ Also **2020 Ext 2 Sample HSC Q9**

9. It is given that a and b are real and p and q are imaginary. **1**

Which pair of inequalities must always be true?

(A) $a^2p^2 + b^2q^2 \leq 2abpq$, $a^2b^2 + p^2q^2 \leq 2abpq$

(B) $a^2p^2 + b^2q^2 \leq 2abpq$, $a^2b^2 + p^2q^2 \geq 2abpq$

(C) $a^2p^2 + b^2q^2 \geq 2abpq$, $a^2b^2 + p^2q^2 \leq 2abpq$

(D) $a^2p^2 + b^2q^2 \geq 2abpq$, $a^2b^2 + p^2q^2 \geq 2abpq$

Question 15

(c) Let n be a positive integer and let x be a positive real number.

i. Show that $x^n - 1 - n(x-1) = (x-1)\left(1 + x + x^2 + \cdots + x^{n-1} - n\right)$ **1**

ii. Hence show that $x^n \geq 1 + n(x-1)$ **2**

iii. Deduce for positive real numbers a and b, **2**

$$a^n b^{1-n} \geq na + (1-n)b$$

A.16 2021 Ext 2 HSC

4. Consider the statement: **1**

'For all integers n, if n is a multiple of 6, then n is a multiple of 2'.

Which of the following is the contrapositive of the statement?

(A) There exists an integer n such that n is a multiple of 6 and not a multiple of 2.

(B) There exists an integer n such that n is a multiple of 2 and not a multiple of 6.

(C) For all integers n, if n is not a multiple of 2, then n is not a multiple of 6.

(D) For all integers n, if n is not a multiple of 6, then n is not a multiple of 2.

5. Which of the following statements is FALSE? **1**

(A) $\forall a, b \in \mathbb{R}, \quad a < b \Rightarrow a^3 < b^3$

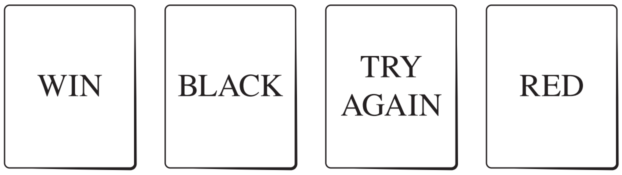
(B) $\forall a, b \in \mathbb{R}, \quad a < b \Rightarrow e^{-a} > e^{-b}$

(C) $\forall a, b \in (0, +\infty), \quad a < b \Rightarrow \ln a < \ln b$

(D) $\forall a, b \in \mathbb{R}$, with $a, b \neq 0, \quad a < b \Rightarrow \dfrac{1}{a} > \dfrac{1}{b}$

9. Four cards have either RED or BLACK on one side and either WIN or TRY AGAIN on the other side. **1**

Sam places the four cards on the table as shown below.



Card 1 Card 2 Card 3 Card 4

A statement is made: 'If a card is RED, then it has WIN written on the other side'.

Sam wants to check if the statement is true by turning over the minimum number of cards.

Which cards should Sam turn over?

(A) 1 and 4 (B) 3 and 4 (C) 1, 2 and 4 (D) 1, 3 and 4

Question 12

(b) Consider Statement A.

Statement A: 'If n^2 is even, then n is even.'

i. What is the converse of Statement A? **1**

ii. Show that the converse of Statement A is true. **1**

Question 15

(a) For all non-negative real numbers x and y,

$$\sqrt{xy} = \frac{x+y}{2} \qquad \text{(Do NOT prove this)}$$

i. Using this fact, show that for all non-negative real numbers a, b and c, **2**

$$\sqrt{abc} \leq \frac{a^2+b^2+2c}{4}$$

ii. Using part (i), or otherwise, show that for all non-negative real numbers a, b and c, **2**

$$\sqrt{abc} \leq \frac{a^2+b^2+c^2+a+b+c}{6}$$

(b) For integers $n \geq 1$, the triangular numbers t_n are defined by $t_n = \dfrac{n(n+1)}{2}$, giving $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$ and so on.

For integers $n \geq 1$, the hexagonal numbers h_n are defined by $h_n = 2n^2 - n$, giving $h_1 = 1$, $h_2 = 6$, $h_3 = 15$, $h_4 = 28$ and so on.

i. Show that the triangular numbers t_1, t_3, t_5, and so on, are also hexagonal numbers. **2**

ii. Show that the triangular numbers t_2, t_4, t_6, and so on, are not hexagonal numbers. **1**

A.17 2022 Ext 2 HSC

2. The following proof aims to establish that $-4 = 0$. **1**

$$\begin{array}{rrl}
\text{Let} & a = -4 & \\
\Rightarrow & a^2 = 16 \text{ and } 4a + 4 = -12 & \text{Line 1} \\
\Rightarrow & a^2 + 4a + 4 = 4 & \text{Line 2} \\
\Rightarrow & (a+2)^2 = 2^2 & \text{Line 3} \\
\Rightarrow & a + 2 = 2 & \text{Line 4} \\
\Rightarrow & a = 0 &
\end{array}$$

At which line is the implication incorrect?

(A) Line 1 (B) Line 2 (C) Line 3 (D) Line 4

3. Let A, B and P be three points in three-dimensional space with $A \neq B$. **1**

Consider the following statement.

If P is on the line AB, then there exists a real number λ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

Which of the following is the contrapositive of this statement?

(A) If for all real numbers λ, $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, then P is on the line AB.

(B) If for all real numbers λ, $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$, then P is not on the line AB.

(C) If there exists a real number λ such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, then P is on the line AB.

(D) If there exists a real number λ such that $\overrightarrow{AP} \neq \lambda\overrightarrow{AB}$, then P is not on the line AB.

7. Consider the statement P. **1**

P : For all integers $n \geq 1$, if n is a prime number then $\dfrac{n(n+1)}{2}$ is a prime number.

Which of the following is true about this statement and its converse?

(A) The statement P and its converse are both true.

(B) The statement P and its converse are both false.

(C) The statement P is true and its converse is false.

(D) The statement P is false and its converse is true.

Question 13

(a) Prove that for all integers n with $n \geq 3$, if $2^n - 1$ is prime, then n cannot be even. **3**

Question 16

(a) It is given that for positive numbers $x_1, x_2, x_3, \cdots, x_n$ with arithmetic mean A,

$$\frac{x_1 \times x_2 \times x_3 \times \cdots \times x_n}{A^n} \leq 1 \qquad \text{(Do NOT prove this)}$$

Suppose a rectangular prism has dimensions a, b and c and surface area S.

i. Show that $abc \leq \left(\dfrac{S}{6}\right)^{\frac{3}{2}}$. **2**

ii. Using part (i), show that when the rectangular prism with surface area S is a cube, it has maximum volume. **2**

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NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \ne 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

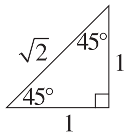
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

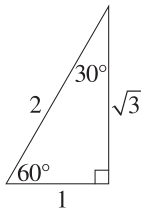
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

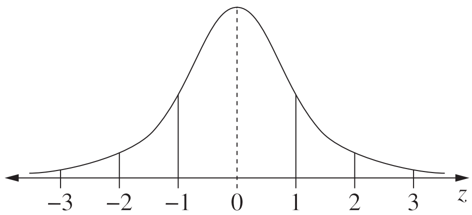
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n - r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

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